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2002 J. Phys. A: Math. Gen. 35 8243

(http://iopscience.iop.org/0305-4470/35/39/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 8243–8251 PII: S0305-4470(02)37057-4

Null generalized helices in Lorentz–Minkowski spaces

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Received 15 May 2002, in final form 4 July 2002
Published 17 September 2002
Online at stacks.iop.org/JPhysA/35/8243

Abstract

We obtain a Lancret-type theorem for null generalized helices in Lorentz–Minkowski spaces $\mathbb{L}^n$. In $\mathbb{L}^3$ we find that the only null generalized helices are the ordinary null helices. However, in $\mathbb{L}^5$ we have to consider two types of null generalized helices according to whether the axis is non-null or null. In both cases we obtain the solutions to the natural equations problem.

PACS number: 02.40.−k
Mathematics Subject Classification: 53B30, 53C50, 53A04

1. Introduction

A generalized helix in $\mathbb{R}^3$ is a curve of constant slope, in other words, a curve whose tangent makes a constant angle with a fixed direction (called the axis). Further, it is a curve whose tangent indicatrix is a planar curve. The study of these curves in $\mathbb{R}^3$ dates from 1802 when Lancret stated that 'a curve is a generalized helix if and only if the ratio of curvature to torsion is constant' (see [15] for the details).

The n-dimensional case (n odd) was considered by Hayden in 1931 (see [10]), who called a generalized helix a curve satisfying that the ratios $\kappa_{2i}/\kappa_{2i-1}$ are constant, $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ being the curvatures of the curve. Hayden proved in [10, 11] that a curve is a generalized helix if there exists a parallel vector field lying in the osculating space of the curve which makes constant angles with the tangent and the principal normals.

The Lancret theorem was revisited and solved by Barros (see [2]) in three-dimensional real space forms by using Killing vector fields along curves. Recently, new improvements have been achieved in Lorentzian space forms. In [3], a non-null curve immersed in $\mathbb{L}^3$ is called a generalized helix if its tangent indicatrix is laid in a plane. Then according to the causal character of this plane the authors have to distinguish between degenerate and non-degenerate generalized helices and show the corresponding Lancret theorem.

To point out the interest of non-null generalized helices it should be mentioned that they arise in the context of the interplay between geometry and integrable Hamiltonian systems (see [12, 13]). In [4], we have found parametrized solutions of the localized induction equation $\partial\gamma/\partial s = \partial\gamma/\partial t \times \partial^2\gamma/\partial t^2$ (LIE) in the three-dimensional Lorentzian space forms, so that the soliton solutions are the null geodesics of the Lorentzian cylinders or *B*-scrolls. Therefore, there is a natural geometric evolution on generalized helices inducing a modified Korteweg–de Vries curvature evolution equation coming from LIE. The role of generalized helices here is probably similar to that of curves of constant torsion or constant natural curvature (see [13]). In [8], we propose the equation $\partial\gamma/\partial s = \partial^2\gamma/\partial t^2 \times \partial^3\gamma/\partial t^3 \times \cdots \times \partial^n\gamma/\partial t^n$ as the corresponding LIE for null curves $\gamma(t) = \gamma(t, 0)$ in the n-dimensional Lorentz–Minkowski space. Then we find that null generalized helices in $\mathbb{L}^n$ evolving in the axis direction are solutions of the null LIE.

Other applications of generalized helices can be found in [14], where the author proposed a mathematical model of the auditory process in the cochlea that does not neglect the effect of cochlea coiling, and [9], where the authors obtained, for inhomogeneous electromagnetic waves in isotropic media, the operator evolution solutions of Maxwell equations; in the case of homogeneous waves an evolution operator is associated with a set of right-handed and left-handed generalized helices.

It is well known that in the geometry of null curves the natural parameter is the pseudo-arc (see [5, 18]). In [7], we generalize the results of Bonnor by introducing a Frenet frame (which we call the Cartan frame) along a null curve with the minimun number of curvature functions and classify null helices. In this paper, we use the Cartan frame (and the Cartan curvatures) introduced there to define and study null generalized helices in $\mathbb{L}^n$ (some results for null curves in three-dimensional spaces are obtained in [1]).

This paper is organized as follows. First, we recall the Cartan frames for null curves in an orientable Lorentzian manifold and obtain similar equations for spacelike curves in lightlike totally geodesic submanifolds in a Lorentzian space form. By using those equations, in the next section we define the null generalized helices in odd-dimensional spaces and obtain a Lancret-type theorem (see theorem 4). In section 4, we find that the only generalized helices in $\mathbb{L}^3$ are the ordinary helices (proposition 5). Furthermore, we get a characterization of null generalized helices in $\mathbb{L}^5$ (theorem 7). From there we can solve the natural equations problem for generalized helices with non-degenerate axis (theorem 8). Finally, the existence and properties of null generalized helices with null axis in $\mathbb{L}^5$ are considered (theorem 9), as well as the solution of the natural equations problem (theorem 10).

2. Frenet equations

Throughout this paper we will follow the notation and terminology stated in [7]. Let M_1^n be an orientable Lorentzian manifold and let $\gamma : I \subset \mathbb{R} \to M_1^n$ be a null curve parametrized by the pseudo-arc. We have the following result.

Theorem 1 [7]. *Assume that* $\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\}$ *is a basis of* $T_{\gamma(t)}M_1^n$ *for all t. Then there exists exactly one Frenet frame* $\{L, W_1, N, W_2, \ldots, W_m\}$, $m = n - 2$, *satisfying the equations*

$$\begin{aligned}
L' &= W_1 \\
W_1' &= -k_1 L + N \\
N' &= -k_1 W_1 + k_2 W_2 \\
W_2' &= k_2 L + k_3 W_3 \\
W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1} \qquad i \in \{3, \ldots, m-1\} \\
W_m' &= -k_m W_{m-1}
\end{aligned} \tag{1}$$

and fulfilling

(i) $\{\gamma', \gamma'', \ldots, \gamma^{(i)}\}$ *and* $\{L, W_1, N, W_2, \ldots, W_{i-2}\}$ *have the same orientation for* $1 \leqslant i \leqslant m-1$,

(ii) $\{L, W_1, N, W_2, \ldots, W_m\}$ *is positively oriented.*

Furthermore, the curvature functions satisfy $k_i > 0$ *for all* $i \geqslant 2$.

The above Frenet frame and curvature functions are called the *Cartan frame* and the *Cartan curvatures* of the null curve γ. In this case γ is called a null *Cartan curve*.

2.1. Frenet frames for spacelike curves in lightlike totally geodesic submanifolds of a Lorentzian space form

Let N^m be a lightlike totally geodesic submanifold in an oriented Lorentzian manifold M_1^n and let $\beta : J \subset \mathbb{R} \to N^m$ be a spacelike curve. Let us assume that $\{\beta'(s), \ldots, \beta^{(m)}(s)\}$ is a basis of $T_{\beta(s)}N^m$, where s stands for the arc-length parameter. Write $E_i(s) = \operatorname{span}\{\beta'(s), \ldots, \beta^{(i)}(s)\}$, for $1 \leqslant i \leqslant m$, and assume that $\dim \operatorname{rad}(E_i(s))$ is constant for all $s \in J$, where $\operatorname{rad}(E_i) = \{\xi \in E_i \mid g(\xi, v) = 0, v \in E_i\}$. Since β is contained in a lightlike submanifold, there exists an index $1 \leqslant i_0 \leqslant m$ such that $\dim \operatorname{rad}\big(E_{i_0}\big) = 1$. Let us denote $r = \min\{i \mid \dim \operatorname{rad}(E_i) = 1\}$, then $r > 1$ because β is spacelike, and $\dim \operatorname{rad}(E_j) = 1$ for $j > r$.

Now we construct a Frenet frame for this kind of curve. The first vector will be $T(s) = \beta'(s)$. As in the non-degenerate case, and using the Gram–Schmidt method applied to E_{r-1}, we can construct a set of orthonormal spacelike vectors $\{T, V_1, \ldots, V_{r-2}\}$ such that $E_{j+1} = \operatorname{span}\{T, V_1, \ldots, V_j\}$, for $1 \leqslant j \leqslant r-2$. Since $\dim \operatorname{rad}(E_r) = 1$, we can find a vector $\mathcal{L}$ (not unique) such that E_r is the orthogonal direct sum of E_{r-1} and $\operatorname{span}\{\mathcal{L}\}$. A straightforward computation shows that the following equations hold:

$$\begin{aligned} T' &= \rho_1 V_1 \\ V_1' &= -\rho_1 T + \rho_2 V_2 \\ V_i' &= -\rho_i V_{i-1} + \rho_{i+1} V_{i+1} \qquad 2 \leqslant i \leqslant r-3 \\ V_{r-2}' &= -\rho_{r-2} V_{r-3} + \rho_{r-1} \mathcal{L} \end{aligned} \tag{2}$$

where $\rho_j : J \to \mathbb{R}$ are differentiable functions and $(\,)'$ denotes a covariant derivative in N^m, which in this case agrees with the covariant derivative of the ambient space M_1^n.

Now it is easy to see that $r = m$ and the Frenet frame is $\{T, V_1, \ldots, V_{m-2}, \mathcal{L}\}$. This frame, as in the non-degenerate case, can be constructed in a unique way (up to orientation) except $\mathcal{L}$. The vector $\mathcal{L}$ can be arbitrarily chosen depending on each situation; a good choice is $\rho_{m-1} = \pm 1$. In any case we need at least $m-1$ curvature functions in order to completely determine the curve. Moreover, the most natural criterion to choose the orientation is achieved by considering that $\{\beta', \ldots, \beta^{(i+1)}\}$ and $\{T, V_1, \ldots, V_i\}$, $1 \leqslant i \leqslant m-2$ have the same orientation, and that $\{T, V_1, \ldots, V_{m-2}, \mathcal{L}\}$ is positively oriented.

The following theorems of existence, uniqueness and congruence can be proved in a similar way as in [7]. Here $M_1^n(c)$ denotes a Lorentzian space form of constant curvature c.

Theorem 2. *Let* $\rho_1, \rho_2, \ldots, \rho_m : [-\varepsilon, \varepsilon] \to \mathbb{R}$ *be differentiable functions. Let* N^m *be a lightlike totally geodesic submanifold of* $M_1^n(c)$, p *a point in* N^m *and consider a positively oriented pseudo-orthonormal basic* $\big\{T^0, V_1^0, \ldots, V_{m-2}^0, \mathcal{L}^0\big\}$ *of* T_pN^m. *Then there exists a unique spacelike Cartan curve* α *in* $M_1^n(c)$, *contained in* N^m *with* $\alpha(0) = p$, *whose Cartan frame* $\{T, V_1, \ldots, V_{m-2}, \mathcal{L}\}$ *satisfies* $T(0) = T^0$, $V_j(0) = V_j^0$ $(j = 1, \ldots, m-2)$ *and* $\mathcal{L}(0) = \mathcal{L}^0$.

Theorem 3. *Let C and $\bar{C}$ be two spacelike Cartan curves which are laid in a lightlike totally geodesic submanifold $N^m \subset M_1^n(c)$, having Cartan curvatures $\{\rho_1, \ldots, \rho_m\}$ in N^m. Then there exists a Lorentzian transformation of M_1^n (or N^m) which maps C into $\bar{C}$.*

3. Null generalized helices in $\mathbb{L}^n$

In a similar way to the non-degenerate case (see [10, 11, 16, 17, 19, 20]), we present the following definition.

Definition 1. *A null Cartan curve $\gamma : I \to \mathbb{L}^n$ $(n = 2q + 3)$ is said to be a generalized helix if there exists a non-zero constant vector v such that all the products $\langle L(t), v\rangle \neq 0$, $\langle N(t), v\rangle$ and $\langle W_{2i+1}(t), v\rangle \neq 0$, $1 \leqslant i \leqslant q - 1$ are constant.*

In the non-degenerate case, the vectors appearing in the definition are unitary and the constancy of the products implies that the curve γ makes a constant angle with some of the vectors of the Cartan frame. The straight line generated by v, which can be spacelike, timelike or lightlike, is uniquely determined and will be called the *axis* of γ. When v is a non-null vector (i.e. spacelike or timelike), we can assume without loss of generality that v is unitary.

A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [15] for the details) says that 'a curve in $\mathbb{R}^3$ is a generalized helix if and only if the ratio of curvature to torsion is constant'. A straightforward computation from theorem 1 yields a generalization of this result for null curves in $\mathbb{L}^n$.

Theorem 4 (the Lancret theorem for null curves)**.** *Let $\gamma : I \to \mathbb{L}^n$ be a null Cartan curve. Then the following statements are equivalent:*

(i) there exist constants $\{r, r_1, \ldots, r_q\}$ $(r_i \neq 0)$ such that

$$k_1(t) = r \qquad \text{and} \qquad k_{2i+1}(t) = r_i k_{2i}(t) \qquad \text{for} \quad 1 \leqslant i \leqslant q \quad n = 2q + 3.$$

(ii) γ is a generalized helix.

Proof. Let us write R_i as the constant $\prod_{j \geqslant i} r_j$ and consider the vector field along the curve γ given by

$$v(t) = R_1(rL + N) + \sum_{i=1}^{q-1} R_{i+1} W_{2i+1} + W_{n-2}.$$

Taking into account (1), statement (i) and the definition of R_i, we have that $v(t)$ is a constant vector. Moreover, it is easy to see that $\langle v, L\rangle$, $\langle v, N\rangle$ and $\langle v, W_{2i+1}\rangle$ are constant, and this concludes the first part of the proof.

Conversely, let v be the axis. Then it is not difficult to see that v is orthogonal to W_{2i+2}, $1 \leqslant i \leqslant q - 1$, so we can write

$$v = \varrho_0 L + \varrho_1 N + \sum_{i=1}^{q} \varrho_{2i+1} W_{2i+1}$$

for certain functions ϱ_j. From the hypothesis we easily deduce that all ϱ_j are constant. Finally, from $v'(t) = 0$ we get

$$k_1 = \frac{\varrho_0}{\varrho_1} = r \qquad k_{2i+1} = \frac{\varrho_{2i-1}}{\varrho_{2i+1}} k_{2i} = r_i k_{2i}$$

and the proof is complete. □

Table 1. Different types of helices, up to congruences, in $\mathbb{L}^3$.

Curve	Curvature	Axis
$\gamma(t) = \left(-\frac{t}{\sigma}, \frac{1}{\sigma^2}\sin\sigma t, \frac{1}{\sigma^2}\cos\sigma t\right)$	$k = \frac{1}{2}\sigma^2 > 0$	$v = (1, 0, 0)$ timelike
$\gamma(t) = \left(\frac{1}{\omega^2}\sinh\omega t, \frac{1}{\omega^2}\cosh\omega t, -\frac{t}{\omega}\right)$	$k = -\frac{1}{2}\omega^2 < 0$	$v = (0, 0, 1)$ spacelike
$\gamma(t) = \left(\frac{t^3}{4} + \frac{t}{3}, \frac{t^2}{2}, \frac{t^3}{4} - \frac{t}{3}\right)$	$k = 0$	$v = (1, 0, 1)$ null

4. Null generalized helices in low dimensions

4.1. The three-dimensional case

A null Cartan curve $\gamma : I \to \mathbb{L}^3$ is a generalized helix if there exists a nonzero constant vector v such that $\langle \gamma', v\rangle$ is constant. This means that the tangent indicatrix is laid in a plane, or in other words, there exists a nonzero vector v in $\mathbb{L}^3$ which is orthogonal to the acceleration vector field of γ. The following result is an easy consequence of theorem 4.

Proposition 5. *Let $\gamma : I \to \mathbb{L}^3$ be a null Cartan curve. Then γ is a generalized helix if and only if γ is a Cartan helix.*

It is well known that, up to congruences, there are exactly three types of helices, according to their curvature function (or axis) (see table 1).

4.2. The five-dimensional case with non-null axis

A null Cartan curve $\gamma : I \to \mathbb{L}^5$ is a generalized helix if there exists a nonzero constant vector v satisfying that $\langle v, L\rangle$ and $\langle v, N\rangle$ are constant. Let v be a unit vector and set $\langle v, v\rangle = \varepsilon$. Let Σ denote the hyperplane orthogonal to v, P the projection map onto the hyperplane Σ and $\bar{\beta} = P(\gamma)$ the projected curve. Then we can write

$$\gamma(t) = \bar{\beta}(t) + \bar{\mu}(t)v \tag{3}$$

where $\bar{\mu} : I \to \mathbb{R}$ is a non-constant differentiable function. By differentiating we have $\langle \bar{\beta}'(t), \bar{\beta}'(t)\rangle = -\bar{\mu}'(t)^2\varepsilon$ so that $\bar{\beta}$ is spacelike (resp. timelike) according to v is timelike (resp. spacelike). Let $\beta : J \to \Sigma$ be the arc-length parametrization of $\bar{\beta}$ with curvature functions $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$. A straightforward computation leads to the following result.

Lemma 6. *Let γ be a null Cartan curve in $\mathbb{L}^5$ and let β be the orthogonal projection of γ onto a non-degenerate hyperplane Σ. Let us denote by t and s the pseudo-arc and arc parameters of γ and β, respectively. Then s and t are linearly related if and only if the first curvature $\tilde{k}_1$ of β is constant. Moreover, in this case, k_1 is constant if and only if $\tilde{k}_2$ is constant.*

We are ready to obtain a relationship between null generalized helices in $\mathbb{L}^5$ with non-degenerate axis and non-degenerate curves in a hyperplane of $\mathbb{L}^5$.

Theorem 7. *Let γ be a null Cartan curve in $\mathbb{L}^5$, v a constant unit vector, Σ the hyperplane orthogonal to v in $\mathbb{L}^5$ and β the projection of γ onto Σ. Then γ is a generalized helix with axis v if and only if β is a curve in Σ with constant curvature and torsion.*

Proof. From lemma 6, to prove the first implication we only need to show that s and t are linearly related. It is not difficult to see that $\lambda t'(s) = \varepsilon\mu'(s) = \varepsilon$, where $\langle L, v\rangle = \lambda$ is constant, so we deduce that $t(s)$ is a linear function.

Conversely, let us assume that $\tilde{k}_1$ and $\tilde{k}_2$ are constant. From lemma 6 we have that t and s are linearly related, and k_1 is constant. A straightforward computation relating the curvatures of γ and β leads to $k_3 = rk_2$ with $r^2 = \tilde{k}_1 \big/ \tilde{k}_2^2$, and this concludes the proof. □

As a consequence, we have

Theorem 8 (solving natural equations for generalized helices with non-degenerate axis)**.** *Let γ be a null Cartan curve in $\mathbb{L}^5$. Then γ is a generalized helix with non-degenerate axis if and only if it is a null geodesic of a Lorentzian cylinder constructed on a non-degenerate curve in $\mathbb{R}^4$ or $\mathbb{L}^4$ with constant curvature and torsion.*

4.3. The five-dimensional case with null axis

From now on, we will deal with null generalized helices with null axis. The main difficulty now arises because the orthogonal hyperplane to the axis is also lightlike and there is no unique way of projection.

Let v be the axis of the helix and let Σ be the orthogonal hyperplane, so that $v \in \Sigma$. From the general theory of lightlike hypersurfaces (see [6]), we have the splitting

$$T_p\mathbb{L}^5 = T_p\Sigma \oplus \mathrm{tr}(T_p\Sigma) = (\mathrm{span}\{v\} \perp S(T_p\Sigma)) \oplus \mathrm{tr}(T_p\Sigma) \qquad \text{for all} \quad p \in \Sigma$$

where $\perp$ denotes orthogonal direct sum, $\mathrm{tr}(T\Sigma) = \cup_{p\in\Sigma} \mathrm{tr}(T_p\Sigma)$ is called a *screen transversal vector bundle* and $S(T\Sigma) = \cup_{p\in\Sigma} S(T_p\Sigma)$ is called a *screen distribution*. Then each choice of a screen distribution provides a projection map on Σ, so that the problem is finding a canonical screen distribution (or the most canonical screen distribution in some sense).

Let $\gamma : I \to \mathbb{L}^5$ be a null generalized helix with the null axis v. Since $\langle L, v\rangle = \lambda$ is constant, then $\tilde{L} = \frac{1}{\lambda}L$ is a transversal section along γ satisfying $\langle \tilde{L}, v\rangle = 1$. Let $\bar{\beta}$ denote the projection of γ with respect to $\tilde{L}$, which is given by $\bar{\beta}(t) = \gamma(t) - \langle\gamma(t), v\rangle\tilde{L}(t)$. Since $\langle\gamma'(t), v\rangle = \lambda$, then $\langle\gamma(t), v\rangle = \lambda(t+\sigma)$ where σ is a constant, from which we have

$$\bar{\beta}(t) = \gamma(t) - (t+\sigma)L(t). \tag{4}$$

The Frenet equations (2) read as follows

$$\begin{aligned} T' &= \rho_1 V_1 \\ V_1' &= -\rho_1 T + \rho_2 V_2 \\ V_2' &= -\rho_2 V_1 + \rho_3 v \\ v' &= 0 \end{aligned} \tag{5}$$

where we have chosen $\mathcal{L} = v$.

Theorem 9. *Let γ be a null Cartan curve in $\mathbb{L}^5$. If γ is a generalized helix with null axis v and Σ denotes its lightlike orthogonal hyperplane, then the curvatures ρ_1, ρ_2 and ρ_3 of the projected spacelike curve $\bar{\beta}$ satisfy*

$$\rho_1(s) = \frac{\tilde{r}}{\sqrt{s}} \qquad \textit{and} \qquad \rho_3 = \tilde{r}_1\rho_2 \tag{6}$$

for certain constants $\tilde{r}$ and $\tilde{r}_1$. Conversely, if $\bar{\beta}$ is a spacelike curve in a lightlike hyperplane of $\mathbb{L}^5$ whose curvatures satisfy (6), then there exists a null generalized helix γ in $\mathbb{L}^5$ whose projection onto Σ is just exactly $\bar{\beta}$.

Proof. Let s denote the arc-length parameter of $\bar{\beta}$, then (4) can be rewritten as

$$\beta(s) = \gamma(t(s)) - (t(s)+\sigma)L(t(s))$$

where t stands for the pseudo-arc parameter of γ. Differentiating and using the Frenet equations we get $T(s) = -(t(s)+\sigma)t'(s)W_1(t(s))$, from which we deduce that $t'(s) = 1/\sqrt{\sigma^2 + 2(s+\omega)}$, for a constant ω. Without loss of generality, let us assume that $\sigma = \omega = 0$. Differentiating again and using that k_1 is constant and $k_3 = r_1 k_2$, with r_1 constant, we deduce

$$\rho_1(s) = \frac{\sqrt{k_1}}{\sqrt{s}} \qquad \rho_2(s) = \frac{k_2(s)}{2\sqrt{k_1}\sqrt{s}} \qquad \text{and} \qquad \rho_3(s) = -\frac{r_1 k_2(s)}{\sqrt{2s}}.$$

Then we can take $\tilde{r} = \sqrt{k_1}$ and $\tilde{r}_1 = -\sqrt{2k_1}r_1$, and this concludes the first part of the proof.

Conversely, let $\beta : J \to \Sigma$ be a spacelike curve in a lightlike hyperplane Σ and put $\Sigma = \text{span}\{v\}^\perp$, where v is a null vector. The Frenet frame (5), with curvatures satisfying (6), can be completed (in a unique way) to a basis of $T_{\beta(s)}\mathbb{L}^5$, for all $s \in J$, by adding a vector field $\mathcal{N}(s)$ along $\beta(s)$ such that $\langle \mathcal{L}, \mathcal{N}\rangle = -1$, $\langle T, \mathcal{N}\rangle = \langle V_1, \mathcal{N}\rangle = \langle V_2, \mathcal{N}\rangle = \langle \mathcal{N}, \mathcal{N}\rangle = 0$. It is easy to see that $\mathcal{N}'(s) = \rho_3(s)V_2(s) = \tilde{r}_1\rho_2(s)V_2(s)$. Then a straightforward computation shows that the curve $\bar{\gamma}$ given by

$$\bar{\gamma}(s) = \beta(s) + \frac{\sqrt{s}}{\tilde{r}}\left(V_1(s) - \frac{\tilde{r}_1}{2}v - \frac{1}{\tilde{r}_1}\mathcal{N}(s)\right)$$

is a null generalized helix with the axis v. Now let t be the pseudo-arc parameter of $\bar{\gamma}$ and put $\bar{\gamma}(s) = \gamma(t(s))$. By differentiating we get $\langle \bar{\gamma}''(s), \bar{\gamma}''(s)\rangle = t'(s)^4 = 1/4s^2$, and so $t'(s) = 1/\sqrt{2s}$. From here, a long and messy computation yields the Frenet frame of $\bar{\gamma}$:

$$\begin{aligned}
L &= \frac{1}{\sqrt{2}\tilde{r}}\left(V_1 - \frac{\tilde{r}_1}{2}v - \frac{1}{\tilde{r}_1}\mathcal{N}\right)\\
W_1 &= -T\\
N &= \frac{\tilde{r}}{\sqrt{2}}\left(V_1 - \frac{\tilde{r}_1}{2}v - \frac{1}{\tilde{r}_1}\mathcal{N}\right)\\
W_2 &= -V_2\\
W_3 &= -\frac{\tilde{r}_1}{2}v + \frac{1}{\tilde{r}_1}\mathcal{N}.
\end{aligned}$$

These equations imply that $\bar{\gamma}$ is a null generalized helix with curvatures given by

$$k_1 = \tilde{r}^2 \qquad k_2 = 2\tilde{r}\sqrt{s}\rho_2 \qquad \text{and} \qquad k_3 = \sqrt{2}\sqrt{s}\rho_2$$

which concludes the proof. □

Now let γ be a null generalized helix with null axis, whose curvatures satisfy $k_1(t) = r$ and $k_3(t) = r_1 k_2(t)$, then the axis is given by

$$v = -\frac{1}{2}\left(rL + N + \frac{1}{r_1}W_3\right) \qquad r = \frac{1}{2r_1^2}.$$

Let us consider a timelike curve $\beta : J \to \mathbb{L}^5$ with Frenet frame $\{\ell, n_1, n_2, n_3, n_4\}$ and curvatures $\{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4\}$, with $\ell(s) = \beta'(s)$, s standing for the arc parameter. Following [11], the curve β is a generalized helix if there exists a nonzero constant vector v such that the products $\langle \beta', v\rangle$ and $\langle n_2, v\rangle$ are constant. In this case, the Lancret theorem assures us that $\tilde{k}_2 = \tilde{r}_1\tilde{k}_1$ and $\tilde{k}_4 = \tilde{r}_3\tilde{k}_3$. Moreover, the axis is given by

$$v = \frac{1}{2}\left(\ell + \frac{1}{\tilde{r}_1}n_2 + \frac{1}{\tilde{r}_1\tilde{r}_3}n_4\right)$$

where $\tilde{r}_1 = \sqrt{1 + 1/\tilde{r}_3^2}$ if the axis v is a null vector.

Let us consider the surface S locally parametrized by $X(s, \omega) = \beta(s) + \omega v$. Then $X_s(s, \omega) = \ell(s)$ and $X_\omega = v$, showing that S is a Lorentzian surface of $\mathbb{L}^5$. The null geodesics of S can be parametrized by

$$\bar{\gamma}(s) = \beta(s) - (s + \sigma)v \tag{7}$$

where σ is a constant. Let t be the pseudo-arc parameter of $\bar{\gamma}$ (as a curve in $\mathbb{L}^5$), then a long and messy computation, from equation (7), yields the following relations among the Cartan curvatures of $\bar{\gamma}$ and the generalized helix β:

$$k_1(t(s)) = \frac{1}{2}\frac{\tilde{k}_1(s)}{\tilde{r}_3^2} - \frac{7}{8}\frac{\tilde{k}_1'(s)^2}{\tilde{k}_1(s)^3} + \frac{1}{2}\frac{\tilde{k}_1''(s)}{\tilde{k}_1(s)^2}$$

$$k_2(t(s)) = \frac{\sqrt{1 + \tilde{r}_3^2}}{\tilde{r}_3}\tilde{k}_3^2(s)$$

$$k_3(t(s)) = \sqrt{\frac{1 + \tilde{r}_3^2}{\tilde{k}_1(s)}}\tilde{k}_3(s).$$

As a consequence, we obtain that $\bar{\gamma}$ is a null generalized helix with null axis if and only if $\tilde{k}_1(s)$ is constant. In this case, $k_1 = r = \tilde{k}_1 / (2\tilde{r}_3^2)$ and $r_1 = \tilde{r}_3/\sqrt{\tilde{k}_1}$. From here and using the theorem of existence and uniqueness of timelike curves in $\mathbb{L}^5$, we can prove the following theorem.

Theorem 10 (solving natural equations for generalized helices with degenerate axis)**.** *Let γ be a null Cartan curve in $\mathbb{L}^5$. Then γ is a generalized helix with null axis if and only if it is a geodesic of a Lorentzian-ruled surface whose directrix is a timelike generalized helix in $\mathbb{L}^5$ (with null axis and constant first curvature) and whose rulings have the direction of the axis.*

Acknowledgments

We wish to thank the referees for their careful reading of the manuscript and constructive comments that have substantially improved the presentation of the paper. This research has been partially supported by Dirección General de Investigación (MCYT) grant BFM2001-2871 through FEDER funds. The second author is supported by a FPPI Grant, Program PG, Ministerio de Educación y Ciencia.

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